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On the hyperspaces $\mathcal{C}_n(X)$ of a continuum X

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Abstract

In 1939 M. Wojdysławski showed that a continuum X is locally connected if and only if for each positive integer n, $\mathcal{C}_n(X)$ is an absolute retract. Since then, nothing else has been done about these hyperspaces. We present some of the properties of these hyperspaces.

Keywords: Absolute retract; Admissible Whitney map; Aposyndetic continuum; Arc-smooth continuum; Colocally connected continuum; Confluent map; Continuum; Decomposable continuum; Finitely aposyndetic continuum; Hyperspace; Indecomposable continuum; Inverse limit; Monotone map; Terminal continuum; Unicoherent continuum; Whitney map

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1. Introduction

In 1939, Wojdysławski proved the following result (see [29, Théorème II_m], where $\mathcal{C}_n(X)$ is denoted by $(2^X)_n$).

Theorem. *A continuum X is locally connected if and only if for each positive integer n, $\mathcal{C}_n(X)$ is an absolute retract.*

Since then, nothing else has been done about these hyperspaces. The purpose of this paper is to present a study of such hyperspaces. The paper is divided in several sections. In Section 2, we give the basic definitions needed for understanding the paper. In Section 3, we present some general properties of the hyperspaces $\mathcal{C}_n(X)$ of a continuum X. In Section 4, it is shown that for any continuum X and any positive integer n, $\mathcal{C}_n(X)$ has trivial shape and

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it is unicoherent. In Section 5, we prove that for any continuum X and any positive integer n, $\mathcal{C}_n(X)$, is finitely aposyndetic. In Section 6, we study when a point in $\mathcal{C}_n(X)$ arcwise disconnects it. In Section 7, admissible Whitney maps are studied. In Section 8, we give some properties of the hyperspace $\mathcal{C}_\infty(X)$. At the end we include some open questions.

2. Definitions

If (Z,d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$, the interior of A is denoted by $Int_Z(A)$, and its closure will be denoted by $\overline{A}$. The symbol $\mathbb{N}$ denotes the set of positive integers.

A *continuum* is a nonempty, compact, connected, metric space. A *subcontinuum* of a space Z is a continuum contained in Z. A continuum is said to be *decomposable* provided it can be written as the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A continuum X is *unicoherent* provided that if $X = A \cup B$, where A and B are proper subcontinua of X, then $A \cap B$ is connected.

An *arc* is any space homeomorphic to $[0,1]$. A *map* means a continuous function.

Given a continuum X, we define its *hyperspaces* as the following sets:

$$
\begin{aligned}
2^X &= \{A \subset X \mid A \text{ is closed and nonempty}\},\\
\mathcal{C}(X) &= \{A \in 2^X \mid A \text{ is a connected}\},\\
\mathcal{C}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ components}\}, \quad n \in \mathbb{N},\\
\mathcal{C}_\infty(X) &= \{A \in 2^X \mid A \text{ has finitely many components}\},\\
\mathcal{F}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}, \quad n \in \mathbb{N},\\
\mathcal{F}(X) &= \{A \in 2^X \mid A \text{ is finite}\}.
\end{aligned}
$$

We agree that $\mathcal{C}(X) = \mathcal{C}_1(X)$. Let us observe that for each $n \in \mathbb{N}$,

$$
\begin{aligned}
&\mathcal{F}_n(X) \subset \mathcal{C}_n(X),\\
&\mathcal{C}_n(X) \subset \mathcal{C}_{n+1}(X),\\
&\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X),\\
&\mathcal{C}_\infty(X) = \bigcup_{n=1}^{\infty} \mathcal{C}_n(X),
\end{aligned}
$$

and that

$$\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X).$$

On the other hand, it is known that 2^X is a metric space with the Hausdorff metric, $\mathcal{H}$, defined as follows:

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\},$$

(see [25, (0.1)]), in fact, 2^X and $\mathcal{C}(X)$ are arcwise connected continua (see [25, (1.13)]), and for each $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are continua (for $\mathcal{C}_n(X)$ see Theorem 1 below, and for $\mathcal{F}_n(X)$ see [4, p. 877]). Hence both $\mathcal{C}_\infty(X)$ and $\mathcal{F}(X)$ are connected subsets of 2^X. On the other hand, 2^X can be topologized with the *Vietoris Topology*, defined as follows: given a finite collection, $U_1, U_2, \ldots, U_m$, of open sets of X, we define

$$\langle U_1, \ldots, U_m\rangle = \left\{ A \in 2^X \;\middle|\; A \subset \bigcup_{k=1}^{m} U_k \text{ and } A \cap U_k \neq \emptyset \text{ for each } k \in \{1, \ldots, m\} \right\}.$$

It is known that the family of all subsets of 2^X of the form $\langle U_1, \ldots, U_m\rangle$, as defined above, form a basis for a topology for 2^X (see [25, (0.11)]) called *Vietoris Topology*, and that the Vietoris Topology and the Topology induced by the Hausdorff metric coincide (see [25, (0.13)]). To simplify notation, $\langle U_1, \ldots, U_m\rangle_n$ denotes the intersection of the open set $\langle U_1, \ldots, U_m\rangle$, of the Vietoris Topology, with $\mathcal{C}_n(X)$.

Let us observe that $\mathcal{F}_1(X)$ is an isometric copy of X contained in each of the above hyperspaces. The hyperspaces $\mathcal{F}_n(X)$ have been extensively studied see: [2,4,5,7,12,14, 17,21–23].

On 2^X, we can define a real-valued function

$$\mu : 2^X \to [0, \infty)$$

called a *Whitney map* which satisfies the following:

(i) μ is continuous;
(ii) for every $x \in X$, $\mu(\{x\}) = 0$;
(iii) if $A \subset B$ and $A \neq B$ then $\mu(A) < \mu(B)$ (see [25, (0.50)]).

Let us observe that if $\nu : 2^X \to [0, \infty)$ is a Whitney map, then we can define another Whitney map, μ, by $\mu(A) = \nu(A)/\nu(X)$. In this case the range of μ is the unit interval $[0, 1]$. Hence, there is no loss of generality in assuming that the range of the Whitney maps is $[0, 1]$.

Given a map $f : X \to Y$ between continua, we consider the induced maps $2^f : 2^X \to 2^Y$, $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y)$, and for each $m \in \mathbb{N}$, $\mathcal{C}_m(f) : \mathcal{C}_m(X) \to \mathcal{C}_m(Y)$, which are given by $2^f(A) = f(A)$, $\mathcal{C}(f)(A) = f(A)$, and $\mathcal{C}_m(f)(A) = f(A)$ for each $m \in \mathbb{N}$. These functions are continuous, and note that $2^f|_{\mathcal{C}(X)} = \mathcal{C}(f)$, and $2^f|_{\mathcal{C}_m(X)} = \mathcal{C}_m(f)$ for each $m \in \mathbb{N}$.

An *inverse sequence* is a "double sequence" $\{X_n, f_n^{n+1}\}$ of spaces X_n and maps $f_n^{n+1} : X_{n+1} \to X_n$. The spaces X_n are called *coordinate spaces* and the maps f_n^{n+1} are called *bonding maps*. Given an inverse sequence $\{X_n, f_n^{n+1}\}$, the *inverse limit of* $\{X_n, f_n^{n+1}\}$, denoted by $\varprojlim\{X_n, f_n^{n+1}\}$, is defined as

$$\varprojlim\left\{X_n, f_n^{n+1}\right\} = \left\{ \{x_n\} \in \prod_{n=1}^{\infty} X_n \;\middle|\; f_n^{n+1}(x_{n+1}) = x_n \text{ for each } n \in \mathbb{N} \right\}.$$

For each $k \in \mathbb{N}$, the map $f_k : \varprojlim\{X_n, f_n^{n+1}\} \to X_k$ will be the restriction of the projection map $\pi_k : \prod_{n=1}^{\infty} X_n \to X_k$ to $\varprojlim\{X_n, f_n^{n+1}\}$.

It is known that if $\{X_n, f_n^{n+1}\}$ is an inverse sequence of continua then its inverse limit, $\varprojlim\{X_n, f_n^{n+1}\}$, is a continuum (see [25, (1.158)]).

Other definitions are given as required.

3. General properties

We begin with some elementary properties of the hyperspaces $\mathcal{C}_n(X)$.

Theorem 3.1. *If X is a continuum and $n \in \mathbb{N}$, then $\mathcal{C}_n(X)$ is an arcwise connected continuum.*

Proof. Let $n \in \mathbb{N}$ be given. Since $\mathcal{C}(X)$ is an arcwise connected continuum (see [25, (1.13)]), we have that $\mathcal{F}_n(\mathcal{C}(X))$ is an arcwise connected continuum, by [4, p. 877] and [7, Lemma 2.2]. Let $\sigma : 2^{2^X} \to 2^X$ be given by $\sigma(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$. It is known that σ is a continuous function (see [25, (1.48)]). Let us observe that $\sigma(\mathcal{F}_n(\mathcal{C}(X))) = \mathcal{C}_n(X)$, thus $\sigma|_{\mathcal{F}_n(\mathcal{C}(X))} : \mathcal{F}_n(\mathcal{C}(X)) \to \mathcal{C}_n(X)$ is an onto map. Therefore, $\mathcal{C}_n(X)$ is an arcwise connected continuum. □

Theorem 3.2. *Let $n \in \mathbb{N}$. Then the continuum X is locally connected if and only if $\mathcal{C}_n(X)$ is locally connected.*

Proof. If X is locally connected, then Wojdysławski has shown that $\mathcal{C}_n(X)$ is an absolute retract (see [29, Théorème II$_m$]), hence locally connected.

Suppose $\mathcal{C}_n(X)$ is locally connected. Let x be a point in X and let U be an open subset of X containing x. Since $\mathcal{C}_n(X)$ is locally connected, there exists a connected open subset $\mathcal{V}$ of $\mathcal{C}_n(X)$ such that $\{x\} \in \mathcal{V} \subset \overline{\mathcal{V}} \subset \langle U \rangle_n$. Let $\langle V_1, \ldots, V_k \rangle_n$ be a basic open set of $\mathcal{C}_n(X)$ such that $\{x\} \in \langle V_1, \ldots, V_k \rangle_n \subset \mathcal{V}$. Let $V = \bigcap_{j=1}^k V_j$. Now, if $y \in V$ then $y \in \sigma(\overline{\mathcal{V}})$ (where σ is defined in the proof of the previous theorem). Since $\{x\} \in \overline{\mathcal{V}}$, we have that $\sigma(\overline{\mathcal{V}}) \in \mathcal{C}(X)$ (see [25, (1.49)]). Thus, both x and y belong to $\sigma(\overline{\mathcal{V}}) \subset U$. Hence X is connected im kleinen at x. Since x was an arbitrary point of X, we have that X is locally connected. □

The proof of the following result is not difficult.

Theorem 3.3. *Let X be a continuum and let $n \in \mathbb{N}$. Then $\mathcal{C}_n(X)$ is nowhere dense in $\mathcal{C}_{n+1}(X)$. In particular $\mathcal{C}_n(X)$ is nowhere dense in 2^X.*

An *order arc* in 2^X is an arc $\alpha : [0,1] \to 2^X$ such that if $0 \leqslant s < t \leqslant 1$ then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$.

Theorem 3.4. *Let X be a continuum and let $n \in \mathbb{N}$. Then $\mathcal{C}_n(X)$ contains an n-cell.*

Proof. Let $A_1, \ldots, A_n$ be n pairwise disjoint nondegenerate subcontinua of X. For each $j \in \{1, \ldots, n\}$, let $a_j \in A_j$, and let $\alpha_j : [0,1] \to \mathcal{C}(X)$ be an order arc such that $\alpha_j(0) = \{a_j\}$ and $\alpha_j(1) = A_j$ (see [25, (1.8)]). Then the map $\xi : [0,1]^n \to \mathcal{C}_n(X)$ given by $\xi(t_1, \ldots, t_n) = \alpha_1(t_1) \cup \cdots \cup \alpha_n(t_n)$ is an embedding of $[0,1]^n$ in $\mathcal{C}_n(X)$. □

If the continuum X contains decomposable continua, more can be said.

Theorem 3.5. *Let X be a continuum and let $n \in \mathbb{N}$. If X contains n pairwise disjoint decomposable subcontinua then $\mathcal{C}_n(X)$ contains a $2n$-cell.*

Proof. Let $M_1, \ldots, M_n$ be n pairwise disjoint decomposable subcontinua of X. Suppose that $M_j = A_j \cup B_j$, where A_j and B_j are continua, for each $j \in \{1, \ldots, n\}$. By the proof of (1.145) of [25], we may assume that each $A_j \cap B_j$ is connected, $A_j \setminus (A_j \cap B_j) \neq \emptyset$, $B_j \setminus (A_j \cap B_j) \neq \emptyset$, and $[A_j \setminus (A_j \cap B_j) \cap B_j \setminus (A_j \cap B_j)] = \emptyset$ for any $j \in \{1, \ldots, n\}$. For each $j \in \{1, \ldots, n\}$, let $\alpha_j : [0, 1] \to \mathcal{C}(A_j)$ and $\beta : [0, 1] \to \mathcal{C}(B_j)$ be order arcs such that $\alpha_j(0) = A_j \cap B_j$, $\alpha_j(1) = A_j$, $\beta_j(0) = A_j \cap B_j$, and $\beta_j(1) = B_j$ (see [25, (1.8)]). Then the map $\xi : [0, 1]^{2n} \to \mathcal{C}_n(X)$ given by $\xi(t_1, \ldots, t_{2n}) = \bigcup_{j=1}^{n}(\alpha_j(t_{2j-1}) \cup \beta_j(t_{2n}))$ is an embedding of $[0, 1]^{2n}$ in $\mathcal{C}_n(X)$. □

Corollary 3.6. *If X is a continuum containing an arc, then $\mathcal{C}_n(X)$ contains a $2n$-cell for each $n \in \mathbb{N}$.*

Theorem 3.7. *Let X be a continuum and let $n \in \mathbb{N}$ be given. Then the following are equivalent.*

(1) 2^X *is contractible.*
(2) $\mathcal{C}_n(X)$ *is contractible.*
(3) $\mathcal{C}(X)$ *is contractible.*

Proof. Suppose 2^X is contractible. Then there exists a map $H' : 2^X \times [0, 1] \to 2^X$ such that for each $A \in 2^X$, $H'(A, 0) = A$ and $H'(A, 1) = X$. Let $H : 2^X \times [0, 1] \to 2^X$ be the *segment homotopy associated* with H' defined by

$$H(A, t) = \bigcup \{H'(A, s) \mid 0 \leqslant s \leqslant t\}.$$

Then H is continuous (see [25, (16.3)]). Observe that for each $A \in 2^X$, $H(A, 0) = A$, $H(A, 1) = X$, and $H(\{A\} \times [0, 1])$ is an order arc from A to X. Note that if $A \in \mathcal{C}_n(X)$ and $B \in H(\{A\} \times [0, 1])$ then $B \in \mathcal{C}_n(X)$ (see [25, (1.8)]). Therefore

$$G = H|_{\mathcal{C}_n(X) \times [0,1]} : \mathcal{C}_n(X) \times [0, 1] \to \mathcal{C}_n(X)$$

is a continuous function such that for each $A \in \mathcal{C}_n(X)$, $G(A, 0) = A$ and $G(A, 1) = X$. Hence $\mathcal{C}_n(X)$ is contractible. A similar argument shows that if $\mathcal{C}_n(X)$ is contractible then $\mathcal{C}(X)$ is contractible. The other implication is contained in the proof of (16.7) of [25]. □

A continuum X is said to have *Kelley's property* provided that given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a, b \in X$, $d(a, b) < \delta$, and $a \in A \in \mathcal{C}(X)$ then there exists $B \in \mathcal{C}(X)$ such that $b \in B$ and $\mathcal{H}(A, B) < \varepsilon$.

Corollary 3.8. *If X is a continuum having Kelley's property, then for each $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ is contractible.*

Proof. If X is a continuum having Kelley's property, then 2^X and $\mathcal{C}(X)$ are contractible (see [25, (16.15)]). Hence the result follows from Theorem 3.7. $\square$

A map $f : X \to Y$ between continua is said to be *monotone* if for each $y \in Y$, $f^{-1}(y)$ is connected. We say that f is *confluent* provided that for each subcontinuum Q of Y and each component C of $f^{-1}(Q)$, $f(C) = Q$.

It is known that there exists a Whitney map defined on 2^A, where A is an arc in the Euclidean plane, which is not monotone (see [25, (14.61)]). Thus, it is important to know when a Whitney map defined on the hyperspace 2^X, of a continuum X, is monotone. To this end, we have the following result (see Theorem 8.9, below, and [15]).

Theorem 3.9. *Let X be a continuum, and let $\mu : 2^X \to [0,1]$ be a Whitney map. Then the following are equivalent*:

(a) *μ is monotone.*
(b) *μ is confluent.*
(c) *$\mathcal{F}_n(X) \cap \mu^{-1}([0,t])$ is connected for every $n \in \mathbb{N}$ and every $t \in [0,1]$.*
(d) *$\mathcal{C}_n(X) \cap \mu^{-1}([0,t])$ is connected for every $n \in \mathbb{N}$ and every $t \in [0,1]$.*

Proof. The equivalence between (a), (b) and (c) is given in [17, Theorem 1.2]. We show that (c) implies (d) and (d) implies (a).

Let $n \in \mathbb{N}$ and suppose that $\mathcal{F}_n(X) \cap \mu^{-1}([0,t])$ is connected. Let $A \in \mathcal{C}_n(X) \cap \mu^{-1}([0,t])$. Then A can be written as the union of its components, say $A = \bigcup_{j=1}^m A_j$, where $m \leqslant n$. For each $j \in \{1,\dots,m\}$, let $a_j \in A_j$. Let $\alpha : [0,1] \to \mathcal{C}_n(X)$ be an order arc from $\{a_1,\dots,a_m\}$ to A. Since for each $r \in [0,1]$, $\alpha(r) \subset A$, we have that $\mu(\alpha(r)) \leqslant \mu(A) \leqslant t$ for every $r \in [0,1]$. Hence $\alpha([0,1]) \subset \mathcal{C}_n(X) \cap \mu^{-1}([0,t])$. Therefore, each element of $\mathcal{C}_n(X) \cap \mu^{-1}([0,t])$ can be joined with $\mathcal{F}_n(X) \cap \mu^{-1}([0,t])$ by an arc. Thus, $\mathcal{C}_n(X) \cap \mu^{-1}([0,t])$ is connected.

Suppose that $\mathcal{C}_n(X) \cap \mu^{-1}([0,t])$ is connected for every $n \in \mathbb{N}$ and every $t \in [0,1]$. Let $t \in [0,1]$. Since $\mathcal{C}_\infty(X) \cap \mu^{-1}([0,t]) = \bigcup_{n=1}^\infty \mathcal{C}_n(X) \cap \mu^{-1}([0,t])$ is dense in $\mu^{-1}([0,t])$, then $\mu^{-1}([0,t])$ is connected. Since $\mu^{-1}([t,1])$ is arcwise connected and 2^X is unicoherent (see [25, (1.176)]), then $\mu^{-1}(t) = \mu^{-1}([0,t]) \cap \mu^{-1}([t,1])$ is connected. $\square$

Corollary 3.10. *Let X be a continuum. If μ is a Whitney map for 2^X then μ is monotone if and only if for each $n \in \mathbb{N}$, $\mu|_{\mathcal{C}_n(X)}$ is monotone.*

Proof. Suppose $\mu : 2^X \to [0,1]$ is monotone. Let $n \in \mathbb{N}$ and let $t \in [0,1]$. Then, by Theorem 3.9, $\mathcal{C}_n(X) \cap \mu^{-1}([0,t])$ is connected. Since $\mathcal{C}_n(X) \cap \mu^{-1}([t,1])$ is arcwise connected,

$$\mathcal{C}_n(X) = \big(\mathcal{C}_n(X) \cap \mu^{-1}([0,t])\big) \cup \big(\mathcal{C}_n(X) \cap \mu^{-1}([t,1])\big)$$

and $\mathcal{C}_n(X)$ is unicoherent (see Theorem 4.8, below), we have that

$$\big(\mu|_{\mathcal{C}_n(X)}\big)^{-1}(t) = \mathcal{C}_n(X) \cap \mu^{-1}(t) = \big(\mathcal{C}_n(X) \cap \mu^{-1}([0,t])\big) \cap \big(\mathcal{C}_n(X) \cap \mu^{-1}([t,1])\big)$$

is connected. Therefore $\mu|_{\mathcal{C}_n(X)}$ is monotone.

Suppose $\mu|_{\mathcal{C}_n(X)}$ is monotone for each $n \in \mathbb{N}$. Let $t \in [0,1]$. Observe that $\mu^{-1}(t) \cap \mathcal{C}_n(X) \subset \mu^{-1}(t) \cap \mathcal{C}_{n+1}(X)$ for each $n \in \mathbb{N}$. Hence $\{\mu^{-1}(t) \cap \mathcal{C}_n(X)\}_{n=1}^{\infty}$ is an increasing sequence of continua such that $\mu^{-1}(t) \cap (\bigcup_{n=1}^{\infty} \mathcal{C}_n(X))$ is dense in $\mu^{-1}(t)$. Therefore $\mu^{-1}(t)$ is connected. $\square$

4. Unicoherence

In this section we will show that for every $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ has trivial shape and it is unicoherent. We begin with some preliminary lemmas.

Lemma 4.1 [25, (1.156)]. *Let $\{X_n, f_n^{n+1}\}$ be an inverse sequence. For each $m \in \mathbb{N}$, let*

$$\mathcal{B}_m(X_n, f_n^{n+1}) = \left\{ f_m^{-1}(U) \mid U \text{ is an open subset of } X_m \right\}.$$

Let

$$\mathcal{B}(X_n, f_n^{n+1}) = \bigcup_{m=1}^{\infty} \mathcal{B}_m\big(X_n, f_n^{n+1}\big).$$

Then, $\mathcal{B}(X_n, f_n^{n+1})$ is a base for the topology for $\varprojlim\{X_n, f_n^{n+1}\}$.

Lemma 4.2 [25, (1.160)]. *Let $\{X_n, f_n^{n+1}\}$ be an inverse sequence of continua and let X_∞ denote $\varprojlim\{X_n, f_n^{n+1}\}$. Let A be a closed subset of X_∞. For each $n \in \mathbb{N}$, let g_n^{n+1} denote the restriction of f_n^{n+1} to $f_{n+1}(A)$. Then $g_n^{n+1}(f_{n+1}(A)) = f_n(A)$ for each $n \in \mathbb{N}$ and, hence $\{f_n(A), g_n^{n+1}\}$ is an inverse sequence with onto bonding maps. Furthermore,*

$$\varprojlim \left\{ f_n(A), g_n^{n+1} \right\} = A = \left(\prod_{n=1}^{\infty} f_n(A) \right) \cap X_\infty,$$

and we consider $\prod_{n=1}^{\infty} f_n(A)$ as being contained in $\prod_{n=1}^{\infty} X_n$ by inclusion.

Theorem 4.3 [11, p. 183]. *Any continuum is homeomorphic to an inverse limit $\varprojlim\{P_n, f_n^{n+1}\}$ where the spaces P_n are compact connected polyhedra.*

One of the important features of inverse limits is that it commutes with the operation of taking hyperspaces. For the statement of this fact let us adopt the following notations. Let X be a continuum and assume that $X = \varprojlim\{X_n, f_n^{n+1}\}$, where each X_n is a continuum. Then we have several *induced* inverse sequences, namely, $\{2^{X_n}, 2^{f_n^{n+1}}\}$, $\{\mathcal{C}(X_n), \mathcal{C}(f_n^{n+1})\}$, and for each $m \in \mathbb{N}$, $\{\mathcal{C}_m(X_n), \mathcal{C}_m(f_n^{n+1})\}$. Let

$$\begin{aligned} 2^X_\infty &= \varprojlim \left\{ 2^{X_n}, 2^{f_n^{n+1}} \right\}, \\ \mathcal{C}^\infty(X) &= \varprojlim \left\{ \mathcal{C}(X_n), \mathcal{C}(f_n^{n+1}) \right\}, \end{aligned}$$

and, for each $m \in \mathbb{N}$,

$$\mathcal{C}_m^\infty = \varprojlim\{\mathcal{C}_m(X_n), \mathcal{C}_m(f_n^{n+1})\}.$$

Theorem 4.4 [25, (1.169)]. *Let X be a continuum and assume that $X = \varprojlim\{X_n, f_n^{n+1}\}$, where each of the spaces is a continuum. Let 2_∞^X and $\mathcal{C}^\infty(X)$ as above. Then 2_∞^X is homeomorphic to 2^X and $\mathcal{C}^\infty(X)$ is homeomorphic to $\mathcal{C}(X)$. Furthermore there is a homeomorphism $h : 2_\infty^X \to 2^X$ such that $h(\mathcal{C}^\infty(X)) = \mathcal{C}(X)$.*

Corollary 4.5. *Let X be a continuum and assume that $X = \varprojlim\{X_n, f_n^{n+1}\}$, where each space X_n is a continuum. For each $m \in \mathbb{N}$, let $\mathcal{C}_m^\infty(X)$ as above. Then $\mathcal{C}_m^\infty(X)$ is homeomorphic to $\mathcal{C}_m(X)$, for every $m \in \mathbb{N}$.*

Proof. The homeomorphism of Theorem 4.4, is defined as follows (see [25, p. 172]): given an element $\{A_n\}_{n=1}^\infty$ of 2_∞^X, then

$$h(\{A_n\}_{n=1}^\infty) = \varprojlim\{A_n, f_n^{n+1}|_{A_{n+1}}\}.$$

Let $m \in \mathbb{N}$ be given. If $\{A_n\}_{n=1}^\infty \in \mathcal{C}_m^\infty(X)$ then $\{A_n, f_n^{n+1}|_{A_{n+1}}\}$ is an inverse sequence of compacta having at most m components, hence by [24, Lemma 1], we have that $\varprojlim\{A_n, f_n^{n+1}|_{A_{n+1}}\}$ has at most m components. Thus $h(\{A_n\}_{n=1}^\infty) = \varprojlim\{A_n, f_n^{n+1}|_{A_{n+1}}\} \in \mathcal{C}_m(X)$. If $A \in \mathcal{C}_m(X)$ then, by Lemma 4.2, we have that $A = \varprojlim\{f_n(A), f_n^{n+1}|_{f_{n+1}(A)}\}$. Thus $\{f_n(A)\}_{n=1}^\infty \in \mathcal{C}_m^\infty(X)$ and $h(\{f_n(A)\}_{n=1}^\infty) = A$. Thus, $h(\mathcal{C}_m^\infty(X)) = \mathcal{C}_m(X)$ and, therefore, $\mathcal{C}_m^\infty(X)$ is homeomorphic to $\mathcal{C}_m(X)$. □

As a consequence of Theorem 4.3, Corollary 4.5, [29, Théorème II$_m$], and [19, 2.1] we have the following result.

Corollary 4.6. *If X is a continuum, then $\mathcal{C}_n(X)$ has trivial shape for each $n \in \mathbb{N}$.*

Lemma 4.7. *If X is a connected metric space which is not unicoherent then there exists a map, $f : X \to \mathcal{S}^1$, from X onto the unit circle $\mathcal{S}^1$ which is not homotopic to a constant map.*

Given a topological space Y, $\check{H}^1(Y; \mathbb{Z})$ will denote the first Čech Cohomology group with integer coefficients.

Theorem 4.8. *If X is a continuum then for every $n \in \mathbb{N}$, each map from $\mathcal{C}_n(X)$ to the unit circle, $\mathcal{S}^1$, is homotopic to a constant map. In particular, we have that $\mathcal{C}_n(X)$ is unicoherent.*

Proof. By Theorem 4.3, X is homeomorphic to the inverse limit of an inverse sequence of compact and connected polyhedra, $\{P_m, f_m^{m+1}\}$. Since each P_m is locally connected, we have, for each $n \in \mathbb{N}$, $\mathcal{C}_n(P_m)$ is a continuum which is an absolute retract (see [29, Théorème II$_m$]), hence, by [3, 2.4, p. 86] and [10, 8.1] each map from $\mathcal{C}_n(P_m)$ into $\mathcal{S}^1$ is homotopic to a constant map, then we obtain that $\check{H}^1(\mathcal{C}_n(P_m); \mathbb{Z}) = 0$ (see [10, 8.1]). By the continuity theorem for Čech Cohomology (see [27, Theorem 7-7]), we have that $\check{H}^1(\mathcal{C}_n(X), \mathbb{Z}) = 0$. Thus, every map from $\mathcal{C}_n(X)$ into $\mathcal{S}^1$ is homotopic to a constant map (see [10, 8.1]). Therefore, by Lemma 4.7, we have that $\mathcal{C}_n(X)$ is unicoherent for each $n \in \mathbb{N}$. $\square$

5. Aposyndesis

In this section we show that for every $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ is finitely aposyndetic (see below).

A continuum is said to be *colocally connected*, provided that each point of it has a local base of open sets whose complements are connected. The continuum X is *aposyndetic* provided that for each pair of points x and y of X, there exists a subcontinuum W of X such that $x \in Int_X(W) \subset W \subset X \setminus \{y\}$. X is *finitely aposyndetic* provided that for each finite subset F of X and point x of X not in F, there exists a subcontinuum W of X such that $x \in Int_X(W) \subset W \subset X \setminus F$.

Theorem 5.1. *Let X be a continuum and let $n \in \mathbb{N}$ be given. Then $\mathcal{C}_n(X)$ is colocally connected.*

Proof. Let A be a point of $\mathcal{C}_n(X)$. We consider two cases.

First assume that $A \in \mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$. Let $\varepsilon > 0$ be given such that $(\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X)) \cap \mathcal{F}_n(X) = \emptyset$. To see that $\mathcal{C}_n(X) \setminus (\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X))$ is connected, let $B \in \mathcal{C}_n(X) \setminus (\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X))$, and let $B_1, \ldots, B_\ell$ be the components of B, then $\ell \leqslant n$. Since $B \in \mathcal{C}_n(X) \setminus (\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X))$, we have that $\mathcal{H}(A, B) \geqslant \varepsilon$. Hence, either $A \not\subset \mathcal{V}_\varepsilon^d(B)$ or $B \not\subset \mathcal{V}_\varepsilon^d(A)$. If $A \not\subset \mathcal{V}_\varepsilon^d(B)$, then there exists a point a in A such that $a \notin \mathcal{V}_\varepsilon^d(B)$. Thus, for every $b \in B$, $d(a, b) \geqslant \varepsilon$. For each $j \in \{1, \ldots, \ell\}$, let $b_j \in B_j$. Let $\alpha : [0, 1] \to \mathcal{C}_n(X)$ be an order arc from $\{b_1, \ldots, b_\ell\}$ to B (see [25, (1.8)]). Since for each $t \in [0, 1]$, $\alpha(t) \subset B$, and for every $b \in B$, $d(a, b) \geqslant \varepsilon$, we have that $\alpha(t) \notin \mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X)$ for any $t \in [0, 1]$. Thus,

$$\alpha\bigl([0, 1]\bigr) \cup \mathcal{F}_n(X) \subset \mathcal{C}_n(X) \setminus \bigl(\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X)\bigr).$$

If $B \not\subset \mathcal{V}_\varepsilon^d(A)$, then there exists a point b in B such that $b \notin \mathcal{V}_\varepsilon^d(A)$. Thus, for every point a of A, $d(b, a) \geqslant \varepsilon$. Without loss of generality we may assume that $b \in B_1$. For each $j \in \{2, \ldots, \ell\}$, let b_j be any point of B_j. Let $\beta : [0, 1] \to \mathcal{C}_n(X)$ be an order arc from $\{b, b_2, \ldots, b_\ell\}$ to B (see [25, (1.8)]). Since for each $t \in [0, 1]$, $b \in \beta(t) \subset B$ and $d(b, a) \geqslant \varepsilon$ for any $a \in A$, we have that $\beta(t) \notin \mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X)$ for any $t \in [0, 1]$. Thus

$$\beta\bigl([0, 1]\bigr) \cup \mathcal{F}_n(X) \subset \mathcal{C}_n(X) \setminus \bigl(\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X)\bigr).$$

Therefore, if $A \in \mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$ and $\varepsilon > 0$ is given such that $(\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X)) \cap \mathcal{F}_n(X) = \emptyset$, then each element B of $\mathcal{C}_n(X) \setminus (\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X))$ can be joined with $\mathcal{F}_n(X)$ by an order arc completely contained in $\mathcal{C}_n(X) \setminus (\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X))$.

Now assume that $A \in \mathcal{F}_n(X)$. Suppose $A = \{a_1, \ldots, a_k\}$ and let $\varepsilon > 0$ be given such that $\mathcal{V}_\varepsilon^d(a_j) \cap \mathcal{V}_\varepsilon^d(a_m) = \emptyset$ if and only if $j \neq m$ and $j, m \in \{1, \ldots, k\}$. Let $B \in \mathcal{C}_n(X) \setminus (\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X))$ and let $B_1, \ldots, B_\ell$ be the components of B, then $\ell \leqslant n$. Hence, $\mathcal{H}(A, B) \geqslant \varepsilon$. Thus, we have that either $A \not\subset \mathcal{V}_\varepsilon^d(B)$ or $B \not\subset \mathcal{V}_\varepsilon^d(A)$.

If $A \not\subset \mathcal{V}_\varepsilon^d(B)$, then there exists a point a in A such that for each point b of B, $d(a, b) \geqslant \varepsilon$. Without loss of generality we may assume that $a = a_1$. Hence $B \cap \mathcal{V}_\varepsilon^d(a_1) = \emptyset$. Let $\alpha : [0, 1] \to \mathcal{C}_n(X)$ be an order arc from B to X. We claim that for each $t \in [0, 1]$, $\mathcal{H}(\alpha(t), A) \geqslant \varepsilon$. To show this, suppose it is not true. Then there exists a point t_0 in $[0, 1]$ such that $\mathcal{H}(\alpha(t_0), A) < \varepsilon$. Thus, we have that $A \subset \mathcal{V}_\varepsilon^d(\alpha(t_0))$ and $\alpha(t_0) \subset \mathcal{V}_\varepsilon^d(A)$. Since $A \subset \mathcal{V}_\varepsilon^d(\alpha(t_0))$, we have that for each $j \in \{1, \ldots, k\}$, $\alpha(t_0) \cap \mathcal{V}_\varepsilon^d(a_j) \neq \emptyset$. On the other hand, since $\alpha(t_0) \subset \mathcal{V}_\varepsilon^d(A) = \bigcup_{j=1}^k \mathcal{V}_\varepsilon^d(a_j)$, and these balls are pairwise disjoint, we have that each component of $\alpha(t_0)$ is contained in one such ball. In particular, there is a component of $\alpha(t_0)$ contained in $\mathcal{V}_\varepsilon^d(a_1)$. Hence $B \cap \mathcal{V}_\varepsilon^d(a_1) \neq \emptyset$, a contradiction. Therefore

$$\alpha\bigl([0, 1]\bigr) \subset \mathcal{C}_n(X) \setminus \bigl(\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X)\bigr).$$

If $B \not\subset \mathcal{V}_\varepsilon^d(A)$, then given an order arc $\beta : [0, 1] \to \mathcal{C}_n(X)$ from B to X, we have that

$$\beta\bigl([0, 1]\bigr) \subset \mathcal{C}_n(X) \setminus \bigl(\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X)\bigr).$$

Therefore if $A \in \mathcal{F}_n(X)$, where $A = \{a_1, \ldots, a_k\}$, and $\varepsilon > 0$ is given such that $\mathcal{V}_\varepsilon(a_j) \cap \mathcal{V}_\varepsilon(a_m) = \emptyset$ if and only if $j \neq m$ and $j, m \in \{1, \ldots, k\}$, then each element $B \in \mathcal{C}_n(X) \setminus (\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X))$ can be joined with $\{X\}$ by an order arc contained in $\mathcal{C}_n(X) \setminus (\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n(X))$. Therefore $\mathcal{C}_n(X)$ is colocally connected. $\square$

Corollary 5.2. *If X is a continuum and $n \in \mathbb{N}$, then $\mathcal{C}_n(X)$ is aposyndetic and finitely aposyndetic.*

Proof. Clearly any colocally connected continuum is aposyndetic. Given a continuum X and a positive integer n, since $\mathcal{C}_n(X)$ is unicoherent (see Theorem 4.8) and aposyndetic, we have that $\mathcal{C}_n(X)$ is finitely aposyndetic (see [1, Corollary 1]). $\square$

6. Arcwise disconnecting

In this section we consider when a point arcwise disconnects $\mathcal{C}_n(X)$.

The following result follows easily using order arcs.

Theorem 6.1. *Let X be a continuum and let $n \in \mathbb{N}$ be given. If A is a proper subcontinuum of X then $\mathcal{C}_n(X) \setminus \mathcal{C}_n(A)$ is arcwise connected.*

The proof of the following theorem is similar to the one given in [25, (11.3)].

Theorem 6.2. *Let X be a continuum and let $n \in \mathbb{N}$ be given. If $A \in \mathcal{C}_n(X)$ is such that $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected then A is connected.*

Theorem 6.3. *A nondegenerate continuum X is indecomposable if and only if for each $n \in \mathbb{N}$, $\mathcal{C}_n(X) \setminus \{X\}$ is not arcwise connected.*

Proof. Suppose X is decomposable. We show that for each $n \in \mathbb{N}$, $\mathcal{C}_n(X) \setminus \{X\}$ is arcwise connected. The proof is done by induction. The result is known for $n = 1$ (see [25, (1.51)]). Let $n = 2$, and let A and B be to elements two of $\mathcal{C}_2(X) \setminus \{X\}$. If both A and B are connected then there exists an arc in $\mathcal{C}(X) \setminus \{X\}$ joining A and B (see [25, (1.51)]). Suppose A has two components, say A_1 and A_2. Since X is decomposable, there exists two proper subcontinua H and K of X such that $X = K \cup H$. We show there exists an arc in $\mathcal{C}_2(X) \setminus \{X\}$ joining A and an element of $\mathcal{C}(X)$. We have to consider several cases.

If either $A \subset H$ or $A \subset K$, then there exists an order arc joining A with H or K (see [25, (1.8)]), and we are done.

If $A_2 = K$ then let $x \in H \cap K$. Let $\alpha : [0, 1] \to \mathcal{C}_2(X)$ be an order arc joining $A_1 \cup \{x\}$ and $A_1 \cup K = A$. Let $\beta : [0, 1] \to \mathcal{C}_2(X)$ be an order arc joining $A_1 \cup \{x\}$ and H. Then $\alpha([0, 1]) \cup \beta([0, 1])$ is contained in $\mathcal{C}_2(X) \setminus \{X\}$ and contains an arc having A and H as its end points.

If $A_1 \subset H$, $A_2 \cap (H \cap K) \neq \emptyset$, and $A_2 \neq K$. Then $H \cup A_2$ is a proper subcontinuum of X, and we can take an order arc joining A to $H \cup A_2$.

If $A_1 \cap (H \cap K) \neq \emptyset$ and $A_2 \cap (H \cap K) \neq \emptyset$ then let $a_j \in A_j \cap H$ for $j \in \{1, 2\}$. Let $\alpha : [0, 1] \to \mathcal{C}_2(X)$ be an order arc joining $\{a_1, a_2\}$ and A and let $\beta : [0, 1] \to \mathcal{C}_2(X)$ be an order arc having $\{a_1, a_2\}$ and H as its end points. Then $\alpha([0, 1]) \cup \beta([0, 1])$ is contained in $\mathcal{C}_2(X) \setminus \{X\}$ and contains an arc joining A and H.

If $A_1 \subset H \setminus K$ and $A_2 \subset K \setminus H$. Then let $x \in H \cap K$. Let $\alpha : [0, 1] \to \mathcal{C}_2(X)$ be an order arc from A to $H \cup A_2$. Let $\beta[0, 1] \to \mathcal{C}_2(X)$ be an order arc from $\{x\} \cup A_2$ to $H \cup A_2$. Let $\gamma : [0, 1] \to \mathcal{C}_2(X)$ be an order arc from $\{x\} \cup A_2$ to K. Then $\alpha([0, 1]) \cup \beta([0, 1]) \cup \gamma([0, 1])$ is contained in $\mathcal{C}_2(X) \setminus \{X\}$ and contains an arc joining A and K. The rest of the cases are similar to the ones treated.

Now, let $n > 2$ and suppose $\mathcal{C}_n(X) \setminus \{X\}$ is arcwise connected. We are going to show that $\mathcal{C}_{n+1}(X) \setminus \{X\}$ is arcwise connected. Let A and B two points in $\mathcal{C}_{n+1}(X) \setminus \{X\}$. If both A and B belong to $\mathcal{C}_n(X)$, by induction hypothesis we can find an arc in $\mathcal{C}_n(X) \setminus \{X\} \subset \mathcal{C}_{n+1}(X) \setminus \{X\}$ having A and B as its end points. Hence, assume that A has $n + 1$ components. Let $A_1, \ldots, A_{n+1}$ be the components of A. Since X is decomposable, there exists two proper subcontinua H and K of it such that $X = H \cup K$. Since $n > 2$, at least two components of A intersect either H or K, suppose that two components of A intersect H. Without loss of generality we may assume that $A_n \cap H \neq \emptyset$ and $A_{n+1} \cap H \neq \emptyset$. For each $j \in \{1, \ldots, n-1\}$, let $a_j \in A_j$ and let $a_n \in A_n \cap H$ and $a_{n+1} \in A_{n+1} \cap H$. Let $\alpha : [0, 1] \to \mathcal{C}_{n+1}(X)$ be an order arc from $\{a_1, \ldots, a_{n+1}\}$ to A (see [25, (1.8)]). Let $\beta : [0, 1] \to \mathcal{C}_2(X)$ be an order arc from $\{a_n, a_{n+1}\}$ to H. Let $\gamma : [0, 1] \to \mathcal{C}_{n+1}(X)$ be given by $\gamma(t) = \{a_1, \ldots, a_{n-1}\} \cup \beta(t)$. Then γ is continuous, $\gamma(0) = \{a_1, \ldots, a_{n-1}\} \cup \beta(0) = \{a_1, \ldots, a_{n+1}\}$ and $\gamma(1) = \{a_1, \ldots, a_{n-1}\} \cup \beta(1) = \{a_1, \ldots, a_{n-1}\} \cup H \in \mathcal{C}_n(X) \setminus \{X\}$. Hence $\alpha([0, 1]) \cup \gamma([0, 1])$ is contained in $\mathcal{C}_{n+1}(X) \setminus \{X\}$ and contains an arc having A and $\gamma(1)$ as its end points. Similarly, if B has $n + 1$ components, we can find an arc in

$\mathcal{C}_{n+1}(X) \setminus \{X\}$ having B and an element of $\mathcal{C}_n(X)$ as its end points. Thus by induction hypothesis, we are done.

The proof of the reverse implication is similar to the one given in [25, (11.4)]. $\square$

Given a continuum X, a subcontinuum E of X is said to be *terminal* provided that if Y is a subcontinuum of X such that $Y \cap E \neq \emptyset$ then either $Y \subset E$ or $E \subset Y$.

Theorem 6.4. *Let X be a continuum, and let E be a nondegenerate proper subcontinuum of X. Consider the following statements*:

(1) *E is a terminal subcontinuum of X.*

(2) *$2^X \setminus \{E\}$ is not arcwise connected.*

(3) *For each $n \in \mathbb{N}$, $\mathcal{C}_n(X) \setminus \{E\}$ is not arcwise connected.*

(4) *$\mathcal{C}(X) \setminus \{E\}$ is not arcwise connected.*

Then, (1) *implies* (2), (3), *and* (4). *Furthermore, if E is decomposable then all four statements are equivalent.*

Proof. The proof of (1) implies (2) and (4) are given in [25, (11.5)]. The proof of (1) implies (3) is similar to the given in [25, (11.5)].

Now suppose E is a decomposable nondegenerate subcontinuum of X. The equivalence between (1), (2), and (4) is given in [25, (11.5)]. Suppose E is not terminal, then $E \neq X$. We show that for each $n \in \mathbb{N}$, $\mathcal{C}_n(X) \setminus \{E\}$ is arcwise connected. This is done by induction.

For $n = 1$ the result is known. Suppose $\mathcal{C}_n(X) \setminus \{E\}$ is arcwise connected. To show that $\mathcal{C}_{n+1}(X) \setminus \{E\}$ is arcwise connected, let A and B be two points in $\mathcal{C}_{n+1}(X) \setminus \{E\}$. If both A and B belong to $\mathcal{C}_{n+1}(X) \setminus \mathcal{C}_{n+1}(E)$, then there exists an arc having A and B as its end points and contained in $\mathcal{C}_{n+1}(X) \setminus \{E\}$ (see Theorem 6.1). If both A and B belong to $\mathcal{C}_{n+1}(E)$ then by Theorem 6.3, there exists an arc joining A and B and contained in $\mathcal{C}_{n+1}(E) \setminus \{E\} \subset \mathcal{C}_{n+1}(X) \setminus \{E\}$. Thus, suppose, without loss of generality, that $A \in \mathcal{C}_{n+1}(E) \setminus \{E\}$ and $B \in \mathcal{C}_{n+1}(X) \setminus \mathcal{C}_n(E)$.

If A has less than $n+1$ components then, by induction hypothesis, there exists an arc in $\mathcal{C}_n(X) \setminus \{E\}$ joining A and X. So assume A has exactly $n+1$ components. Since E is decomposable, by Theorem 6.3, there exists an arc joining A and an element A' of $\mathcal{C}_n(E) \setminus \{E\}$. By induction hypothesis, there exists an arc in $\mathcal{C}_n(X) \setminus \{E\}$ joining A and X. Hence, there exists an arc in $\mathcal{C}_{n+1}(X) \setminus \{E\}$ joining A and X. Since $B \in \mathcal{C}_{n+1}(X) \setminus \mathcal{C}_{n+1}(E)$, by Theorem 6.1, there exits an arc having B and X as its end points.

Therefore, there exists an arc in $\mathcal{C}_{n+1}(X) \setminus \{E\}$ joining A and B. $\square$

The proof of the following result is similar to the one given in [25, (11.6)].

Corollary 6.5. *If E is a decomposable subcontinuum of a continuum X such that for each $n \in \mathbb{N}$, $\mathcal{C}_n(X) \setminus \{E\}$ is not arcwise connected, then E is nowhere dense in any subcontinuum which properly contains E.*

Theorem 6.6. *Let X be a continuum. Then for any $E \in 2^X$, the following are equivalent*:

(1) *$2^X \setminus \{E\}$ is not arcwise connected.*

(2) *$\mathcal{C}(X) \setminus \{E\}$ is not arcwise connected.*
(3) *For each $n \in \mathbb{N}$, $\mathcal{C}_n(X) \setminus \{E\}$ is not arcwise connected.*

Proof. The proof of the equivalence of (1) and (2) is given in [25, (11.8)]. Clearly (3) implies (2). We show that (2) implies (3).

Let $E \in 2^X$ and suppose $\mathcal{C}(X) \setminus \{E\}$ is not arcwise connected. Then $E \in \mathcal{C}(X)$ (see [25, (11.3)]). Let $n > 1$. If $E = X$, then X is indecomposable (see [25, (1.51)]), and then by Theorem 6.3, $\mathcal{C}_n(X) \setminus \{E\}$ is not arcwise connected.

Suppose $E \neq X$. If E is decomposable then E is a terminal subcontinuum of X (see [25, (11.5)]). Hence by Theorem 6.4, $\mathcal{C}_n(X) \setminus \{E\}$ is not arcwise connected.

Thus, assume E is indecomposable. Let $B \in \mathcal{C}(E) \subset \mathcal{C}_n(X)$ and $A \in \mathcal{C}_n(X) \setminus \mathcal{C}_n(E)$. Let $\alpha : [0,1] \to \mathcal{C}_n(X)$ be an arc such that $\alpha(0) = B$ and $\alpha(1) = A$. Let $\beta : [0,1] \to \mathcal{C}_n(X)$ be given by $\beta(t) = \bigcup \alpha([0,t])$. Then β is an order arc from B to $\bigcup \alpha([0,1])$. Hence, $\beta([0,1]) \subset \mathcal{C}(X)$ (see [25, (1.11)]), in particular $\bigcup \alpha([0,1])$ is connected. Since $\beta(0) = B \subset E$, $\beta(1) = \bigcup \alpha([0,1])$, $(\bigcup \alpha([0,1])) \cap (\mathcal{C}(X) \setminus \mathcal{C}(E)) \neq \emptyset$, and $\mathcal{C}(X) \setminus \{E\}$ is not arcwise connected, we have that there exists $t_0 \in [0,1]$ such that $\beta(t_0) = E$. Let $t_1 = \min\{t \in [0,1] \mid \beta(t) = E\}$. Then $t_1 > 0$ and $\beta(t_1) = E$. Since for each $t < t_1$, $\beta(t)$ is a proper subcontinuum of E and E is indecomposable, we have that $\beta(t)$ is nowhere dense in E (see [20, Theorem 2, p. 207]). Note that for every $t < t_1$, $E = \beta(t) \cup (\bigcup \alpha([t,t_1])$, and $\bigcup \alpha([t,t_1])$ is compact. Hence $E = \bigcup \alpha([t,t_1])$. By continuity, we have that $E = \alpha(t_1)$. Therefore, $\mathcal{C}_n(X) \setminus \{E\}$ is not arcwise connected. □

Theorem 6.7. *Let E, A and B be subcontinua of the continuum X and let $n \in \mathbb{N}$ be given. Then the following are equivalent*:
(1) *If Γ is an arc in $\mathcal{C}(X)$ such that $A, B \in \Gamma$, then $E \in \Gamma$.*
(2) *If Γ is an arc in $\mathcal{C}_n(X)$ such that $A, B \in \Gamma$, then $E \in \Gamma$.*
(3) *If Γ is an arc in 2^X such that $A, B \in \Gamma$, then $E \in \Gamma$.*

Proof. Clearly (3) implies (2) and (2) implies (1). The proof of the implication from (1) to (3) is in [25, (11.13)]. □

Corollary 6.8. *Let E be a subcontinuum of the continuum X. If Λ is an arc component of $2^X \setminus \{E\}$ and $\Lambda \cap \mathcal{C}_n(X) \neq \emptyset$, for some $n \in \mathbb{N}$, then $\Lambda \cap \mathcal{C}_n(X)$ is an arc component of $\mathcal{C}_n(X) \setminus \{E\}$.*

Theorem 6.9. *If X is a continuum then the following are equivalent*:
(1) *X is hereditarily indecomposable.*
(2) *For each nondegenerate subcontinuum E of X, $2^X \setminus \{E\}$ is not arcwise connected.*
(3) *For each nondegenerate subcontinuum E of X, $\mathcal{C}_n(X) \setminus \{E\}$ is not arcwise connected, for each $n \in \mathbb{N}$.*
(4) *For each nondegenerate subcontinuum E of X, $\mathcal{C}(X) \setminus \{E\}$ is not arcwise connected.*

Proof. By Theorem 6.6, we have that (2), (3), and (4) are equivalent. The proof of the equivalence between (4) and (1) is given in [25, (11.15)]. □

7. Admissible Whitney maps

In this section the structure of $\mathcal{C}_n(X)$ is studied under the existence of admissible Whitney maps (see below). The proofs of most of the results presented are similar to the ones given in the indicated references. When needed, the appropriate comments are presented.

A *free arc* in a continuum X is an arc A in X such that $Int_X(A) \neq \emptyset$. A *Hilbert cube* is any space homeomorphic to $\prod_{k=1}^{\infty}[0,1]_k$ with the product topology, where $[0,1]_k = [0,1]$ for each $k \in \mathbb{N}$. A closed subset Z of a continuum X is said to be a Z*-set* provided that for each $\varepsilon > 0$, there exists a map $f : X \to (X \setminus Z)$ such that $d(x, f(x)) < \varepsilon$ for each $x \in X$. A CE *map* is a continuous function whose point inverses all have trivial shape.

In [9, 3.2 and 4.1] Curtis and Schori proved that for any locally connected continuum X, 2^X is a Hilbert cube and, if X contains no free arc, $\mathcal{C}(X)$ is a Hilbert cube. Our first theorem of this section is the analogue result for $\mathcal{C}_n(X)$.

Theorem 7.1. *If X is a locally connected continuum without free arcs then $\mathcal{C}_n(X)$ is homeomorphic to the Hilbert cube for each $n \in \mathbb{N}$.*

Proof. The map g defined in Lemma 4 of [8] can be used to show that, under our hypothesis, $\{X\}$ is a Z-set in $\mathcal{C}_n(X)$. Hence, by Corollary 5.1 of [6], we have that $\mathcal{C}_n(X)$ is homeomorphic to the Hilbert cube. □

Lemma 7.2. *Let $n \in \mathbb{N}$. If X is a continuum and $\mathcal{A} \in \mathcal{C}(\mathcal{C}_n(X))$ then $\bigcup_{A \in \mathcal{A}} A \in \mathcal{C}_n(X)$.*

Proof. Let $\mathcal{A} \in \mathcal{C}(\mathcal{C}_n(X))$ and suppose that $\bigcup_{A \in \mathcal{A}} A \notin \mathcal{C}_n(X)$. Hence, $\bigcup_{A \in \mathcal{A}} A$ has at least $n+1$ components. Thus, we can find $n+1$ nonempty mutually disjoint closed subsets, $C_1, \ldots, C_{n+1}$, of $\bigcup_{A \in \mathcal{A}} A$ such that $\bigcup_{A \in \mathcal{A}} A = \bigcup_{j=1}^{n+1} C_j$. Let

$$\mathcal{B} = \left\{ A \in \mathcal{A} \;\middle|\; A \subset \bigcup_{j=1}^{n} C_j \right\}$$

and

$$\mathcal{D} = \left\{ A \in \mathcal{A} \mid A \cap C_{n+1} \neq \emptyset \right\}.$$

Then $\mathcal{B}$ and $\mathcal{D}$ are disjoint closed subsets of $\mathcal{A}$ and $\mathcal{A} = \mathcal{B} \cup \mathcal{D}$, a contradiction.

Therefore, $\bigcup_{A \in \mathcal{A}} A \in \mathcal{C}_n(X)$. □

Let $n \in \mathbb{N}$, and let X be a continuum. A Whitney map μ for $\mathcal{C}_n(X)$ is called *admissible Whitney map* provided that there is a (continuous) homotopy $H : \mathcal{C}_n(X) \times [0,1] \to \mathcal{C}_n(X)$ satisfying the following two conditions:

(i) for all $A \in \mathcal{C}_n(X)$, $H(A,1) = A$ and $H(A,0) \in \mathcal{F}_1(X)$;

(ii) if $\mu(H(A,t)) > 0$ for some $A \in \mathcal{C}_n(X)$ and some $t \in [0,1]$, then $\mu(H(A,s)) < \mu(H(A,t))$ whenever $0 \leqslant s < t$.

A homotopy $H : \mathcal{C}_n(X) \times [0,1] \to \mathcal{C}_n(X)$ satisfying (i) and (ii) is called a *μ-admissible deformation for $\mathcal{C}_n(X)$*. We remark that it is not asked in (i) that $H(A,0)$ be the same singleton for all $A \in \mathcal{C}_n(X)$.

Theorem 7.3. *Let $n \in \mathbb{N}$ and let X be a continuum. If μ is an admissible Whitney map for $\mathcal{C}_n(X)$ then*:

(1) *If X' is homeomorphic to X then there is an admissible Whitney map for $\mathcal{C}_n(X')$.*
(2) *X is arcwise connected.*
(3) *X is contractible if and only if $\mathcal{C}_n(X)$ is contractible.*
(4) *If X is locally connected then X is contractible.*
(5) *If X is an absolute neighborhood retract then X is an absolute retract.*
(6) *For each $t_0 \in (0, \mu(X))$, $\mu^{-1}(t_0)$ is a retract of $\mu^{-1}([t_0, \mu(X)])$.*
(7) *If $\mathcal{C}_n(X)$ is contractible then $\mu^{-1}(t_0)$ is contractible for each $t_0 \in [0, \mu(X)]$.*
(8) *If X is locally connected, then $\mu^{-1}(t_0)$ is an absolute retract for each $t_0 \in (0, \mu(X))$.*
(9) *For each $t_0 \in (0, \mu(X))$, $\mu^{-1}(t_0)$ has all those properties which are common to all r-images of all hyperspaces. In particular, $\mu^{-1}(t_0)$ is an arcwise connected continuum which has trivial shape* (*and, thus, is acyclic*).
(10) *X has trivial shape.*
(11) *μ is an open CE map.*

Proof. The proofs of (1), (2), (6), (7), (8), and (11) are similar to the ones given in (2.2), (2.3), (2.7), (2.8), (2.9), and (2.12) of [13], respectively. (3) follows from Corollary 3.10 and [13, (2.4)]. (4) follows from Theorem 3.2 and [25, (16.18)]. (5) follows from (4) and [3, 9.1, p. 96]. (9) follows from [13, (2.10)] but Lemma 7.2 has to be used to ensure that the map f defined in [13, (2.10)] sends $\mathcal{C}(\Gamma)$ into Γ. (10) follows from [13, (2.11)] but Corollary 4.6 needs to be used to show that $\mu^{-1}([0,t])$ has trivial shape. $\square$

Theorem 7.4 [13, (2.13)]. *Let X be a continuum with metric d such that there is a homotopy $\xi : X \times [0,1] \to X$ satisfying the following conditions*:

(1) *$\xi(x,1) = x$ and, for some given point $p \in X$, $\xi(x,0) = p$ for all $x \in X$;*
(2) *if $d(\xi(x_1,t), \xi(x_2,t)) > 0$ for some points $x_1, x_2 \in X$ and number $t \in [0,1]$, then $d(\xi(x_1,s), \xi(x_2,s)) < d(\xi(x_1,t), \xi(x_2,t))$ whenever $0 \leqslant s < t$.*

Then, there are admissible Whitney maps for $\mathcal{C}_n(X)$ for each $n \in \mathbb{N}$.

A subset Z of a Banach space E is said to be *starshaped* provided that there is a point $p \in Z$ such that for any $z \in Z$, the convex arc in E from p to z lies in Z.

Theorem 7.5 [13, (2.14)]. *If X is a compact starshaped subset of a Banach space, then there are admissible Whitney maps for $\mathcal{C}_n(X)$ for each $n \in \mathbb{N}$.*

Theorem 7.6 [13, (2.15)]. *If X is the* (*topological*) *cone over a nonempty compact metric space Y, then there are admissible Whitney maps for $\mathcal{C}_n(X)$ for each $n \in \mathbb{N}$.*

Lemma 7.7 [13, (3.2)]. *Let $n \in \mathbb{N}$, let X be a locally connected continuum and let A be a closed subset of X such that $Int_X(A) \neq \emptyset$. Assume that A does not contain any free arc in X. If μ is an admissible Whitney map for $\mathcal{C}_n(X)$, then, for each $t_0 \in (0, \mu(X))$, $\{B \in \mu^{-1}(t_0) \mid A \subset B\}$ is a Z-set in $\mu^{-1}(t_0)$.*

Theorem 7.8 [13, (4.1)]. *Let $n \in \mathbb{N}$, and let X be a locally connected continuum containing no free arc. If there is an admissible Whitney map, μ, for $\mathcal{C}_n(X)$ then $\mu^{-1}(t_0)$ is a Hilbert cube whenever $t_0 \in (0, \mu(X))$.*

As a corollary of Theorems 7.6 and 7.8 we have the following result (compare with [13, (4.2)]).

Theorem 7.9. *Let $n \in \mathbb{N}$. If X is the (topological) cone over any nonempty compact, locally connected metric space Y having nondegenerate components, then there is a Whitney map μ for $\mathcal{C}_n(X)$ such that $\mu^{-1}(t_0)$ is a Hilbert cube for each $t_0 \in (0, \mu(X))$.*

As a consequence of Theorems 7.3(1), 7.5, and 7.8 we have the following (compare with [13, (4.6)]).

Theorem 7.10. *Let $n \in \mathbb{N}$. If X is a locally connected continuum having no free arc and X can be embedded in a Banach space so as to be starshaped, then there is an admissible Whitney map μ for $\mathcal{C}_n(X)$ such that $\mu^{-1}(t_0)$ is a Hilbert cube for each $t_0 \in (0, \mu(X))$.*

Theorem 7.11. *Let $n \in \mathbb{N}$, and let X be a locally connected continuum having no free arcs. If there is an admissible Whitney map μ for $\mathcal{C}_n(X)$ then $\mu^{-1}([0, t_0])$ and $\mu^{-1}([t_0, \mu(X)])$ are Hilbert cubes for every $t_0 \in (0, \mu(X))$.*

Proof. The theorem follows from [13, (4.10)] but Theorem 7.1 needs to be used to ensure that $\mathcal{C}_n(X)$ is a Hilbert cube. □

A map $f : Y \to Z$ between compact metric spaces is a *strongly regular map* if for each $z_0 \in Z$ and every $\varepsilon > 0$ there exists a neighborhood U of z_0 in Z such that if $z \in U$ then there exist maps $g_1 : f^{-1}(z) \to f^{-1}(z_0)$ and $g_2 : f^{-1}(z_0) \to f^{-1}(z)$ such that g_1 and g_2 move points no more that ε and $g_1 \circ g_2$ and $g_2 \circ g_1$ are homotopic to the identity maps on $f^{-1}(z_0)$ and $f^{-1}(z)$, respectively, via homotopies which move points no more than ε. Note that strongly regular maps are open.

Theorem 7.12. *Let $n \in \mathbb{N}$, and let X be a locally connected continuum. If μ is an admissible Whitney map for $\mathcal{C}_n(X)$ then $\mu|_{\mu^{-1}((0,\mu(X)])} : \mu^{-1}((0, \mu(X)]) \to (0, \mu(X)]$ is a strongly regular map whose fibres are absolute retracts.*

Proof. The proof is similar to the one given in [18, (2.5)], but we have to use Theorem 7.3 part (8) to ensure that for each $t \in (0, \mu(X)]$, $\mu^{-1}(t)$ is an absolute retract. □

A map $p : E \to B$ is a *trivial bundle map with F fibres* if there exists a homeomorphism $h : E \to B \times F$ such that $\pi \circ h = p$, where $\pi : B \times E \to B$ is the projection map.

Theorem 7.13. *Let $n \in \mathbb{N}$, and let X be a locally connected continuum containing no free arc. If μ is an admissible Whitney map for $\mathcal{C}_n(X)$ then $\mu|_{\mu^{-1}((0,\mu(X)))} : \mu^{-1}((0, \mu(X))) \to (0, \mu(X))$ is a trivial bundle with Hilbert cube fibres.*

Proof. The proof is similar to the one given in [18, (3.1) (i)]. We have to use Theorem 7.8 to ensure that for each $t \in (0, \mu(X))$, $\mu^{-1}(t)$ is homeomorphic to the Hilbert cube. □

A continuum X is *arc-smooth at a point p* provided that there exists a map $\Delta : X \to \mathcal{C}(X)$ satisfying

(a) $\Delta(p) = \{p\}$.

(b) For each $x \in X \setminus \{p\}$, $\Delta(x)$ is an arc from p to x.

(c) If $y \in \Delta(x)$ then $\Delta(y) \subset \Delta(x)$.

In Theorem 7.4 it was given sufficient conditions for the existence of admissible Whitney maps. A. Illanes showed the existence of admissible Whitney maps for the hyperspaces 2^X and $\mathcal{C}(X)$ of an arc-smooth continuum X. His proof also shows the following (see [16, Theorem 1.3]).

Theorem 7.14. *If X is an arc-smooth continuum at a point p then there exists an admissible Whitney map μ for $\mathcal{C}_n(X)$ for each $n \in \mathbb{N}$.*

8. The hyperspace $\mathcal{C}_\infty(X)$

In this section we prove some general properties of $\mathcal{C}_\infty(X)$.

Theorem 8.1. *Let X be a continuum. Then, $\mathcal{C}_\infty(X)$ is not a G_δ subset of 2^X.*

Proof. Suppose $\mathcal{C}_\infty(X)$ is a G_δ subset of 2^X. Then $\mathcal{C}_\infty(X) = \bigcap_{k=1}^\infty \mathcal{U}_k$. Hence

$$2^X \setminus \mathcal{C}_\infty(X) = 2^X \setminus \bigcap_{k=1}^{\infty} \mathcal{U}_k = \bigcup_{k=1}^{\infty} (2^X \setminus \mathcal{U}_k),$$

and then,

$$\begin{aligned} 2^X &= \mathcal{C}_\infty(X) \cup \left(\bigcup_{k=1}^{\infty} (2^X \setminus \mathcal{U}_k) \right) \\ &= \left(\bigcup_{n=1}^{\infty} \mathcal{C}_n(X) \right) \cup \left(\bigcup_{k=1}^{\infty} (2^X \setminus \mathcal{U}_k) \right). \end{aligned} \tag{$*$}$$

On the other hand, since $\mathcal{C}_\infty(X)$ is dense in 2^X and for each $k \in \mathbb{N}$, $\mathcal{C}_\infty(X) \cap (2^X \setminus \mathcal{U}_k) = \emptyset$, we have that for every $k \in \mathbb{N}$, $Int_{2^X}(2^X \setminus \mathcal{U}_k) = \emptyset$, i.e., $2^X \setminus \mathcal{U}_k$ is nowhere dense for each $k \in \mathbb{N}$. By ($*$) and Theorem 3.3, 2^X can be written as the union of a countable family of closed nowhere dense subsets of it. This contradicts the Baire Category Theorem. Therefore $\mathcal{C}_\infty(X)$ is not a G_δ subset of 2^X. $\square$

Corollary 8.2. *Let X be a continuum. Then $\mathcal{C}_\infty(X)$ is not completely metrizable.*

Proof. It is a consequence of Theorem 8.1 and [28, 24.12]. $\square$

Theorem 8.3. *Let X be a continuum. Then $\mathcal{C}_\infty(X)$ is not locally compact.*

Proof. Suppose $\mathcal{C}_\infty(X)$ is locally compact. Let $B \in \mathcal{C}_\infty(X)$ and suppose $B_1, \dots, B_n$ are the components of B, where $n > 1$. Let $\mathcal{B}$ be a compact neighborhood of B in $\mathcal{C}_\infty(X)$. Let $\varepsilon > 0$ such that $\mathcal{V}_\varepsilon^{\mathcal{C}_\infty(X)}(B) \subset \mathcal{B}$, and $\mathcal{V}_\varepsilon^X(B_j) \cap \mathcal{V}_\varepsilon^X(B_k) = \emptyset$ for each $j \neq k$ and $j, k \in \{1, \dots, n\}$. Let $C \in \mathcal{C}(X)$ be such that $B_1 \subset C \subset \mathcal{V}_\varepsilon^X(B_1)$ and $B_1 \neq C$ (see [26, 5.5]). Let $\{x_\ell\}_{\ell=1}^\infty$ be a sequence of points in $C \setminus B$ converging to a point x in B_1. For each $\ell \in \mathbb{N}$, let $A_\ell = \{x_1, \dots, x_\ell\}$ and let $A_0 = \{x_\ell\}_{\ell=1}^\infty \cup \{x\}$. Note that for each $\ell \in \mathbb{N}$, $B \cup A_\ell \subset \mathcal{V}_\varepsilon^{\mathcal{C}_\infty(X)}(B)$, and the sequence $\{B \cup A_\ell\}_{\ell=1}^\infty$ converges to $B \cup A_0$. But $B \cup A_0 \notin \mathcal{C}_\infty(X)$, a contradiction. Therefore, $\mathcal{C}_\infty(X)$ is not locally compact. $\square$

The following theorem is an easy consequence of Theorem 3.1.

Theorem 8.4. *For any continuum X, $\mathcal{C}_\infty(X)$ is arcwise connected.*

Let us observe that the proof of Theorem 5.1 also shows:

Theorem 8.5. *If X is a continuum, then $\mathcal{C}_\infty(X)$ is colocally connected.*

Theorem 8.6. *A continuum X is locally connected if and only if $\mathcal{C}_\infty(X)$ is locally connected.*

Proof. Suppose X is locally connected. Let $A \in \mathcal{C}_\infty(X)$. If $A = X$ then, for any given $\varepsilon > 0$, it is easy to see that $\mathcal{V}_\varepsilon^{\mathcal{H}}(X) \cap \mathcal{C}_\infty(X)$ is arcwise connected. Thus, we may assume that $A \neq X$. Let $A_1, \dots, A_n$ be the components of A. Let $\mathcal{U} = \langle U_1, \dots, U_n \rangle \cap \mathcal{C}_\infty(X)$ be a basic open subset of $\mathcal{C}_\infty(X)$ such that the sets $U_1, \dots, U_n$ are open, connected, pairwise disjoint and $A_j \subset U_j$ for each $j \in \{1, \dots, n\}$. For each $j \in \{1, \dots, n\}$, choose an open connected subset V_j of X such that $A_j \subset V_j \subset Cl_X(V_j) \subset U_j$. Then $\mathcal{V} = \langle V_1, \dots, V_n \rangle \cap \mathcal{C}_\infty(X)$ is a neighborhood of A. Given $B \in \mathcal{V}$, there exists an order arc $\alpha : [0, 1] \to \mathcal{C}_\infty(X)$ such that $\alpha(0) = B$ and $\alpha(1) = Cl_X(V_1) \cup \dots \cup Cl_X(V_n)$ (see [25, (1.8)]). Notice that for each $t \in [0, 1]$, $\alpha(t) \in \mathcal{U}$. Therefore, $\mathcal{C}_\infty(X)$ is connected in kleinen at A. Since A was arbitrary, $\mathcal{C}_\infty(X)$ is locally connected.

The proof of the other implication is similar to the one given in Theorem 3.2. $\square$

The proof of the following theorem is similar to the one given in Theorem 3.7.

Theorem 8.7. *Let X be a continuum. Then the following are equivalent*:

(1) 2^X *is contractible.*

(2) $\mathcal{C}_\infty(X)$ *is contractible.*
(3) $\mathcal{C}(X)$ *is contractible.*

The proof of the following result is similar to the one given in [17, Lemma1.3].

Theorem 8.8. *If X is a locally connected continuum, then $\mathcal{C}_\infty(X)$ is unicoherent.*

Theorem 8.9. *If X is a locally connected continuum and $\mu : 2^X \to [0,1]$ is a Whitney map, then μ is monotone if and only if $(\mu|_{\mathcal{C}_\infty(X)})^{-1}(t)$ is connected for each $t \in [0,1]$.*

Proof. Suppose that μ is monotone. Let $t \in [0,1]$. By Theorem 3.9, $\mathcal{C}_n(X) \cap \mu^{-1}([0,t])$ is connected for each $n \in \mathbb{N}$. Then

$$\mathcal{C}_\infty(X) \cap \mu^{-1}\big([0,t]\big) = \bigcup_{n=1}^{\infty} \big(\mathcal{C}_n(X) \cap \mu^{-1}([0,t])\big)$$

is connected. Since $\mathcal{C}_\infty(X) \cap \mu^{-1}([t,1])$ is arcwise connected. Then, by Theorem 8.8, we have that $\mathcal{C}_\infty(X) \cap \mu^{-1}(t) = \mu^{-1}_{\mathcal{C}_\infty(X)}(t)$ is connected. The proof of the other implication is similar to the one given in [17, Theorem 1.4]. □

Questions. If X is a continuum and $n \in \mathbb{N}$, then the following are natural questions:
(1) Is $\mathcal{C}_n(X)$ contable closed aposyndetic?
(2) Is $C_n(X)$ 0-dimensional closed aposyndetic?
(3) If $C_n(X)$ does not contain $(n+1)$-cells, then is X hereditarily indecomposable?
(4) Is $C_\infty(X)$ always unicoherent?

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